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Trajectory Optimization Using Nonsingular Orbital Elements and True Longitude

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The system and adjoint differential equations needed to solve low-thrust transfer and rendezvous trajectories are derived in terms of nonsingular orbit elements, with the true longitude selected as the sixth variable that also defines the radial distance. The perturbation accelerations are resolved in the rotating polar frame such that the equations of motion are written in polar coordinates. This formulation is particularly convenient for the treatment of the J_2 perturbation acceleration whose components are easily expressed in terms of the true longitude. The iterations needed for the solution of Kepler's equation are also eliminated, and the analytic form of the adjoint equations is provided in their simplest aspect thus far.

I. Introduction

THE use of orbital elements and in particular nonsingular equinoctial elements has been of great benefit in developing simulation software for trajectory propagation and optimization, especially for low-thrust applications.[1–5] Rapid numerical integration of the equations for the state and adjoint variables is made possible by the averaging technique, which is applicable only if these variables vary slowly in time, precluding thereby the adoption of the Cartesian coordinates preferred by flight mechanics specialists.

There are many ways of writing the equations of motion and their associated adjoint equations because the sixth state variable can be chosen to be the mean longitude, the eccentric longitude, or the true longitude at either the current time or epoch time. This is identical to the adoption of the mean anomaly, the eccentric anomaly, or the true anomaly when the more familiar classical orbit elements are used. The epoch formulations are fundamental in the sense that all of the state variables are truly constant in the absence of any perturbative acceleration, whether it is conservative, such as in the case of the geopotential or third body effects, or nonconservative, such as in the case of thrust or atmospheric drag. The component resolution of these accelerations can be made in various rotating orbital frames, such as the Euler–Hill or polar frame, the equinoctial frame, or even the tangential frame. Furthermore, the right-hand sides of all of these system and adjoint differential equations are written in terms of either the eccentric longitude or the true longitude but not the mean longitude (the radial distance cannot be written in terms of the mean longitude), the equations of motion being defined by the partials of the radial distance with respect to the orbital elements of choice. Various sets of these elements were adopted, coupled with the selection of the equinoctial frame and with the radial distance expressed in terms of the eccentric longitude, to solve the low-thrust transfer problem with only the thrust perturbation acceleration active.[6–8]

In Ref. 9, it has been recognized that the treatment of the J_2 oblateness effect in the context of exact precision-integrated trajectories is made easier by the use of the true longitude in the right-hand side of all of our differential equations because it is more complicated to generate the analytic form of the adjoint equations if the J_2 accelerations are written in terms of the eccentric longitude. This paper is an extension of the analysis presented in Ref. 9 and makes possible a further simplification by adopting the true longitude as the sixth state variable instead of the mean longitude of Ref. 9. The need to solve Kepler's transcendental equation is thus eliminated, and the adjoint equations are obtained in their simplest form thus far. Much of the original theory of optimal low-thrust transfer using equinoctial elements developed by Edelbaum et al., as well as numerous solved transfers, is found in Ref. 10.

II. Equations of Motion with the True Longitude as the Sixth State Variable

The variational equations for the classical orbit elements and in the Gaussian form, with the disturbing acceleration resolved in the rotating Euler–Hill orbital frame (also known as the polar coordinate frame), are given by

$$\dot{a} = (2a^2/h')[es_{\theta^*} f_r + (p'/r) f_\theta] \tag{1}$$

$$\dot{e} = (1/h')\{p' s_{\theta^*} f_r + [(p'+r)c_{\theta^*} + re] f_\theta\} \tag{2}$$

$$\dot{i} = (r c_\theta / h') f_h \tag{3}$$

$$\dot{\Omega} = (r s_\theta / h' s_i) f_h \tag{4}$$

$$\dot{\omega} = \frac{1}{h'e}[-p' c_{\theta^*} f_r + (p'+r) s_{\theta^*} f_\theta] - \frac{r s_\theta c_i}{h' s_i} f_h \tag{5}$$

$$\dot{M} = n + \frac{(1-e^2)^{\frac{1}{2}}}{h'e}[(p' c_{\theta^*} - 2re) f_r - (p'+r) s_{\theta^*} f_\theta] \tag{6}$$

Here $\theta = \omega + \theta^*$ is the orbital angular position, where θ^* is the true anomaly; n is the orbit mean motion; $p' = a(1-e^2)$ is the orbit parameter; $h' = [\mu a(1-e^2)]^{1/2}$ is the orbit angular momentum with μ the Earth gravity constant; and f_r, f_θ, and f_h are the components of the disturbing acceleration vector $\boldsymbol{\Gamma} = \boldsymbol{f}/m = f_t\, \hat{u} = (f_r, f_\theta, f_h)$ with magnitude f_t such that m is the spacecraft mass and f is the magnitude of the disturbing force or, in our case, the thrust vector $\boldsymbol{f}$. The direction of the acceleration vector is given by the unit vector $\hat{u}$ such that $f_r = f_t u_r$, $f_\theta = f_t u_\theta$, and $f_h = f_t u_h$. The radial distance $r = p'/(1 + e c_{\theta^*})$, with c_{θ^*} standing for $\cos\theta^*$ and s_{θ^*} standing for $\sin\theta^*$, etc. Equation (6) for the mean anomaly can be replaced by Eq. (7) for the true anomaly,

$$\dot{\theta}^* = (h'/r^2) + (1/eh')[p' c_{\theta^*} f_r - (p'+r) s_{\theta^*} f_\theta] \tag{7}$$

These equations can also be written for the nonsingular equinoctial elements a, h, k, p, q, and λ, where $a = a$, $h = e\sin(\omega+\Omega)$, $k = e\cos(\omega+\Omega)$, $p = \tan(i/2)\sin\Omega$, $q = \tan(i/2)\cos\Omega$, and $\lambda = M + \omega + \Omega$, which is also called the mean longitude:

$$\dot{a} = \frac{2}{n(1-h^2-k^2)^{\frac{1}{2}}}[(k s_L - h c_L) f_r + (1 + h s_L + k c_L) f_\theta] \tag{8}$$

$$\dot{h} = \frac{(1-h^2-k^2)^{\frac{1}{2}}}{na(1 + h s_L + k c_L)}\{-(1 + h s_L + k c_L) c_L f_r + [h + (2 + h s_L + k c_L) s_L] f_\theta - k(p c_L - q s_L) f_h\} \tag{9}$$

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$$\dot{k} = \frac{(1-h^2-k^2)^{\frac{1}{2}}}{na(1+hs_L+kc_L)}\{(1+hs_L+kc_L)s_L f_r + [k+(2+hs_L+kc_L)c_L]f_\theta + h(pc_L - qs_L)f_h\} \tag{10}$$

$$\dot{p} = \frac{(1-h^2-k^2)^{\frac{1}{2}}}{2na(1+hs_L+kc_L)}(1+p^2+q^2)s_L f_h \tag{11}$$

$$\dot{q} = \frac{(1-h^2-k^2)^{\frac{1}{2}}}{2na(1+hs_L+kc_L)}(1+p^2+q^2)c_L f_h \tag{12}$$

$$\dot{\lambda} = n - \frac{(1-h^2-k^2)^{\frac{1}{2}}}{na(1+hs_L+kc_L)}\Big\{\Big[\beta(1+hs_L+kc_L) \times (hs_L+kc_L) + 2(1-h^2-k^2)^{\frac{1}{2}}\Big]f_r + \beta(2+hs_L+kc_L)(hc_L-ks_L)f_\theta + (pc_L - qs_L)f_h\Big\} \tag{13}$$

The true longitude L appearing in the right-hand side of the preceding differential equations is defined as $L = \theta^* + \omega + \Omega$ or $L = \theta + \Omega$, and the factor β is a function only of h and k with

$$\beta = \frac{1}{1+(1-h^2-k^2)^{\frac{1}{2}}} \tag{14}$$

The preceding equations have been transformed to their present form by making use of the identities $r = a(1-h^2-k^2)/(1+hs_L+kc_L)$, $h' = na^2(1-h^2-k^2)^{1/2}$, and

$$s_{\theta^*} = (a/er)\big[kc_F + hs_F - e^2\big] = -(1/e) + (a/er)(1-e^2) = (1/e)[(p'/r)-1]$$

$$c_{\theta^*} = (a/er)(1-e^2)^{\frac{1}{2}}(ks_F - hc_F)$$

which are then written in terms of L with the use of the identity

$$(ks_F - hc_F) = \frac{r(ks_L - hc_L)}{a(1-h^2-k^2)^{\frac{1}{2}}}$$

We thus have $s_{\theta^*} = (ks_L - hc_L)/(h^2+k^2)^{1/2}$ and $c_{\theta^*} = (hs_L + kc_L)/(h^2+k^2)^{1/2}$, and from $\theta = L - \Omega$, we have $s_\theta = s_L c_\Omega - c_L s_\Omega$ and $c_\theta = c_L c_\Omega + s_L s_\Omega$, which are written in their final form by replacing $s_\Omega = p/(p^2+q^2)^{1/2}$ and $c_\Omega = q/(p^2+q^2)^{1/2}$:

$$s_\theta = \frac{qs_L}{(p^2+q^2)^{\frac{1}{2}}} - \frac{pc_L}{(p^2+q^2)^{\frac{1}{2}}}$$

$$c_\theta = \frac{qc_L}{(p^2+q^2)^{\frac{1}{2}}} + \frac{ps_L}{(p^2+q^2)^{\frac{1}{2}}}$$

We also have $p'/r = (1+hs_L+kc_L)$ and $r/h' = (1-h^2-k^2)^{1/2}/[na(1+hs_L+kc_L)]$.

Because λ is being integrated, the current value of L must be computed from the identities[9]

$$c_L = (a/r)\big[(1-h^2\beta)c_F + hk\beta s_F - k\big] \tag{15}$$

$$s_L = (a/r)\big[hk\beta c_F + (1-k^2\beta)s_F - h\big] \tag{16}$$

after evaluating the eccentric longitude F from Kepler's equation $\lambda = F - ks_F + hc_F$. By definition, $F = E + \omega + \Omega$, where E is the classical eccentric anomaly. We can avoid these intermediate calculations concerning F if we replace Eq. (13) for $\dot{\lambda}$ by the corresponding equation for $\dot{L}$ because we have $\dot{L} = \dot{\theta}^* + (\dot{\omega} + \dot{\Omega}) = \dot{\theta}^* + \dot{\tilde{\omega}}$, with $\tilde{\omega} = \omega + \Omega$ representing the longitude of pericenter. Making use of Eq. (7) and with $h'/r^2 = n(1+hs_L+kc_L)^2/(1-h^2-k^2)^{3/2}$,

$$\dot{\tilde{\omega}} = (1/h'e)[-p'c_{\theta^*}f_r + (p'+r)s_{\theta^*}f_\theta] + (r/h')s_\theta \tan(i/2)f_h \tag{17}$$

we get

$$\dot{L} = \dot{\theta}^* + \dot{\tilde{\omega}} = (h'/r^2) + (r/h')s_\theta \tan(i/2)f_h \tag{18}$$

The term $s_\theta \tan i$ is evaluated from the definitions $s_{i/2} = (p^2+q^2)^{1/2}/(1+p^2+q^2)^{1/2}$ and $c_{i/2} = 1/(1+p^2+q^2)^{1/2}$, such that $\tan i/2 = (p^2+q^2)^{1/2}$ and finally $s_\theta \tan i/2 = (qs_L - pc_L)$. The definitions of p and q also provide directly the preceding expression for $\tan i/2$. We can now write the differential equation for L as

$$\dot{L} = \frac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2} + \frac{r(qs_L - pc_L)}{na^2(1-h^2-k^2)^{\frac{1}{2}}}f_h \tag{19}$$

or

$$\dot{L} = \frac{n(1+hs_L+kc_L)^2}{(1-h^2-k^2)^{\frac{3}{2}}} + \frac{(1-h^2-k^2)^{\frac{1}{2}}}{na(1+hs_L+kc_L)}(qs_L - pc_L)f_h \tag{20}$$

The full set of the equations of motion is now given by Eqs. (8–12) and (19) or (20) with the true longitude L obtained directly through integration. This set will provide us with the simplest form of the adjoint equations because now we can set $\partial L/\partial a = \partial L/\partial h = \partial L/\partial k = 0$ when generating these adjoint differential equations and eliminate the need for solving the transcendental Kepler equation for F through iteration. Let $z = (a\ h\ k\ p\ q\ L)^T$ stand for the state vector with corresponding adjoint $\boldsymbol{\lambda}_z = (\lambda_a\ \lambda_h\ \lambda_k\ \lambda_p\ \lambda_q\ \lambda_L)^T$. The dynamic equations are given in compact form by

$$\begin{pmatrix} \dot{a} \\ \dot{h} \\ \dot{k} \\ \dot{p} \\ \dot{q} \\ \dot{L} \end{pmatrix} = \begin{pmatrix} B_{11}^L & B_{12}^L & B_{13}^L \\ B_{21}^L & B_{22}^L & B_{23}^L \\ B_{31}^L & B_{32}^L & B_{33}^L \\ B_{41}^L & B_{42}^L & B_{43}^L \\ B_{51}^L & B_{52}^L & B_{53}^L \\ B_{61}^L & B_{62}^L & B_{63}^L \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \\ u_h \end{pmatrix} f_t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2} \end{bmatrix} \tag{21}$$

The Hamiltonian and the differential equations for the adjoints are obtained from

$$H = \boldsymbol{\lambda}_z^T B^L(z) f_t \hat{u} + \lambda_L \frac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2} \tag{22}$$

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial z} = -\boldsymbol{\lambda}_z^T \frac{\partial B^L}{\partial z} f_t \hat{u} - \lambda_L \frac{\partial}{\partial z}\left[\frac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2}\right] \tag{23}$$

where B^L is the 6×3 matrix shown in Eq. (21). Because L is now independent of a, h, k, p, and q, we have in Eq. (23) $\partial L/\partial a = \partial L/\partial h = \partial L/\partial k = \partial L/\partial p = \partial L/\partial q = 0$. The radial distance r appears in the matrix B^L such that we need the partials $\partial r/\partial a$, $\partial r/\partial h$, $\partial r/\partial k$, and $\partial r/\partial L$ to generate Eq. (23). The partials $\partial r/\partial p = \partial r/\partial q = 0$ because r is an in-plane variable, as is also L.

From the definition of r,

$$r = \frac{a(1-h^2-k^2)}{1+hs_L+kc_L} \tag{24}$$

it is straightforward to get

$$\frac{\partial r}{\partial a} = \frac{r}{a} \tag{25}$$

We also have

$$\frac{\partial r}{\partial h} = \frac{-2ah}{1+hs_L+kc_L} - \frac{a(1-h^2-k^2)s_L}{(1+hs_L+kc_L)^2} \tag{26}$$

which can also be written as

$$\frac{\partial r}{\partial h} = -\frac{r}{a(1-h^2-k^2)}(2ah + rs_L) \tag{27}$$

In a similar way, we have the following two forms for $\partial r/\partial k$:

$$\frac{\partial r}{\partial k} = \frac{-2ak}{1 + hs_L + kc_L} - \frac{a(1-h^2-k^2)c_L}{(1+hs_L+kc_L)^2} \tag{28}$$

$$\frac{\partial r}{\partial k} = -\frac{r}{a(1-h^2-k^2)}(2ak + rc_L) \tag{29}$$

Finally,

$$\frac{\partial r}{\partial L} = -\frac{r^2}{a}\frac{hc_L - ks_L}{1-h^2-k^2} \tag{30}$$

This partial can also be written in terms of the eccentric longitude F because we have the identity[9]

$$ks_F - hc_F = \frac{r}{a}\frac{ks_L - hc_L}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{31}$$

such that

$$\frac{\partial r}{\partial L} = \frac{r(ks_F - hc_F)}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{32}$$

It is also shown in Ref. 9 that

$$\frac{\partial L}{\partial F} = \frac{a}{r}(1-h^2-k^2)^{\frac{1}{2}} = \frac{1+hs_L+kc_L}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{33}$$

which is obtained by direct differentiation of the identities $c_L = (a/r)\,[(1-h^2\beta)c_F + hk\beta s_F - k]$ and $s_L = (a/r)\,[hk\beta c_F + (1-k^2\beta)s_F - h]$.

We can now verify that

$$\frac{\partial r}{\partial F} = \frac{\partial r}{\partial L}\frac{\partial L}{\partial F} = \frac{r(ks_F - hc_F)}{(1-h^2-k^2)^{\frac{1}{2}}} \times \frac{a}{r}(1-h^2-k^2)^{\frac{1}{2}} = a(ks_F - hc_F) \tag{34}$$

which is the same expression obtained directly from the definition of r in terms of F, namely, $r = a(1 - kc_F - hs_F)$ and therefore $\partial r/\partial F = a(ks_F - hc_F)$. It is also shown in Ref. 9 that $\partial r/\partial\lambda = (a^2/r^2)(1-h^2-k^2)^{1/2}$ such that the chain rule

$$\frac{\partial r}{\partial\lambda} = \frac{\partial r}{\partial L}\frac{\partial L}{\partial\lambda} = \frac{a(ks_L - hc_L)}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{35}$$

recovers the expression for the partial $\partial r/\partial\lambda$ obtained in Ref. 9.

Let us now digress momentarily from the task of deriving Eq. (23) and discuss the similarities between the formulation developed in this paper and the formulation based on the set (a, h, k, p, q, F) developed in Ref. 8. Neither of these two formulations requires solution of Kepler's transcendental equation iteratively at each integration step because the dynamic equations for the set (a, h, k, p, q, L) are written in terms of the sixth variable L, and the dynamic equations for the set (a, h, k, p, q, F) are written in terms of the sixth variable F. The fact that the polar frame is used in the present case and the equinoctial frame in the second case, for the component resolution of the disturbing acceleration, is irrelevant to our discussion. In both of these formulations, $\partial r/\partial a = r/a$. However, if we compare $\partial r/\partial h = -as_F$ of Ref. 8 with $\partial r/\partial h$ of Eq. (26) or (27), we will find, after expressing $-as_F$ in terms of L, that these two partials, as expected, are not identical. The same holds true for the $\partial r/\partial k$ partial of Ref. 8, which is equal to $-ac_F$, and the present partial in Eq. (28) or (29). The expressions for s_F and c_F in terms of L are

$$s_F = h + \frac{r}{a}\frac{(1-h^2\beta)s_L - hk\beta c_L}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{36}$$

$$c_F = k + \frac{r}{a}\frac{(1-k^2\beta)c_L - hk\beta s_L}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{37}$$

Let us now go back to the original formulations of Refs. 7 and 9, which use the set (a, h, k, p, q, λ). In Ref. 9, L appears as the accessory variable in the right-hand side of the equations of motion. Therefore $L = f(h, k, F)$ with $F = f(h, k, \lambda)$ such that

$$\left(\frac{\partial r}{\partial h}\right)_{\text{tot}} = \left(\frac{\partial r}{\partial h}\right)_L + \frac{\partial r}{\partial L}\left(\frac{\partial L}{\partial h}\right)_{\text{tot}} = \left(\frac{\partial r}{\partial h}\right)_L + \frac{\partial r}{\partial L}\left[\left(\frac{\partial L}{\partial h}\right)_F + \frac{\partial L}{\partial F}\frac{\partial F}{\partial h}\right] \tag{38}$$

On the other hand, from Ref. 7, we have with $r = a(1 - kc_F - hs_F)$,

$$\left(\frac{\partial r}{\partial h}\right)_{\text{tot}} = \left(\frac{\partial r}{\partial h}\right)_F + \frac{\partial r}{\partial F}\frac{\partial F}{\partial h} \tag{39}$$

However, $\partial r/\partial F$ can be written as

$$\frac{\partial r}{\partial F} = \frac{\partial r}{\partial L}\frac{\partial L}{\partial F} \tag{40}$$

such that Eq. (39) takes the following form:

$$\left(\frac{\partial r}{\partial h}\right)_{\text{tot}} = \left(\frac{\partial r}{\partial h}\right)_F + \frac{\partial r}{\partial L}\frac{\partial L}{\partial F}\frac{\partial F}{\partial h} \tag{41}$$

Equations (38) and (41) must be identical, which in turn shows that

$$\left(\frac{\partial r}{\partial h}\right)_L + \left(\frac{\partial r}{\partial L}\right)\left(\frac{\partial L}{\partial h}\right)_F = \left(\frac{\partial r}{\partial h}\right)_F \tag{42}$$

Because $(\partial r/\partial h)_F = -as_F$,

$$\frac{\partial r}{\partial L} = -\frac{r^2}{a}\frac{hc_L - ks_L}{1-h^2-k^2} = \frac{r(ks_F - hc_F)}{(1-h^2-k^2)^{\frac{1}{2}}}$$

and because, from Ref. 9, $\partial L/\partial F = (a/r)(1-h^2-k^2)^{1/2}$ with $\partial F/\partial h = -(a/r)c_F$, we can now evaluate $(\partial L/\partial h)_F$ needed in Eq. (42) by making use of the $(\partial L/\partial h)_{\text{tot}}$ partial developed in Ref. 9 such that, in terms of F,

$$\left(\frac{\partial L}{\partial h}\right)_{\text{tot}} = \frac{-(a^2/r^2)(s_F - h)(1-h^2-k^2) + (Y_1/a) + 2h}{(1-h^2-k^2)^{\frac{1}{2}}(ks_F - hc_F)} = \left(\frac{\partial L}{\partial h}\right)_F + \frac{\partial L}{\partial F}\frac{\partial F}{\partial h} \tag{43}$$

Therefore, with $(\partial L/\partial h)_{\text{tot}}$ given by Eq. (43),

$$\left(\frac{\partial L}{\partial h}\right)_F = \left(\frac{\partial L}{\partial h}\right)_{\text{tot}} - \frac{\partial L}{\partial F}\frac{\partial F}{\partial h} = \left(\frac{\partial L}{\partial h}\right)_{\text{tot}} + \frac{a^2}{r^2}(1-h^2-k^2)^{\frac{1}{2}}c_F$$

which is now used in Eq. (42) with $(\partial r/\partial h)_L$ given by Eq. (27). Because $Y_1 = rs_L$, the left-hand side of Eq. (42) can be written as

$$-\frac{2hr}{1-h^2-k^2} - \frac{rY_1}{a(1-h^2-k^2)} - \frac{a^2}{r}(s_F - h) + \frac{rY_1}{a(1-h^2-k^2)} + \frac{2hr}{1-h^2-k^2} + \frac{a^2}{r}(ks_F - hc_F)c_F$$
$$= -\frac{a^2}{r}s_F + \frac{a^2}{r}h\left(1 - c_F^2\right) + \frac{a^2}{r}ks_Fc_F = -\frac{a^2}{r}(1 - hs_F - kc_F)s_F = -as_F$$

which is the expression of $(\partial r/\partial h)_F$. Therefore, Eq. (42) holds, showing how $(\partial r/\partial h)_L$ differs from $(\partial r/\partial h)_F$.

Identical arguments also show how $(\partial r/\partial k)_L$ differs from $(\partial r/\partial k)_F$. These manipulations show that, when either L or F is selected as the sixth state variable, the variation of r due to the variations in h and k must not account for the variation of r due to the variations of the respective anomalies. Furthermore, holding L fixed or F fixed will yield different partials, as was just shown. When the orbit is circular, $e = 0$ or $h = k = 0$ and $r = a$, and the expression for $\partial r/\partial h$ in Eq. (27) reduces to $\partial r/\partial h = -rs_L$. However, from Eq. (36) we also get $s_F = s_L$ such that $\partial r/\partial h = -as_F$ and therefore $(\partial r/\partial h)_L = (\partial r/\partial h)_F$. When $e \neq 0$, these two partials are therefore not identical. The same observations in Eqs. (29) and (37) show that $(\partial r/\partial k)_L = -rc_L$ with $c_L = c_F$ when $e = 0$, and in that case $(\partial r/\partial k)_L = (\partial r/\partial k)_F$, too. We can now derive the $\partial B^L/\partial z$ partials. We have with $\partial r/\partial h$, $\partial r/\partial k$, and $\partial r/\partial L$ given by Eqs. (27), (29), and (30), respectively, and with $\partial n/\partial a = -3n/2a$, and letting $G = (1 - h^2 - k^2)^{1/2}$ and $K = (1 + p^2 + q^2)$, the elements of the B^L matrix and their partial derivatives with respect to all six orbital elements as given in the Appendix.

III. Numerical Results

Let us now establish the mathematical relationships between the Lagrange multipliers of the (a, h, k, p, q, λ) formulation of Ref. 9 and the present (a, h, k, p, q, L) formulation. Using the λ superscript and the L superscript for the λ and L formulations, respectively, we can write

$$\lambda_a^\lambda\, da + \lambda_h^\lambda\, dh + \lambda_k^\lambda\, dk + \lambda_p^\lambda\, dp + \lambda_q^\lambda\, dq + \lambda_\lambda^\lambda\, d\lambda + H^\lambda\, dt$$
$$= \lambda_a^L\, da + \lambda_h^L\, dh + \lambda_k^L\, dk + \lambda_p^L\, dp$$
$$+ \lambda_q^L\, dq + \lambda_L^L\, dL + H^L\, dt \tag{44}$$

The Hamiltonians are both constant because both systems are autonomous and equal to the unit value to satisfy the transversality condition for a minimum-time transfer. We now need to express dL in terms of dh, dk, and $d\lambda$. From $L = f(h, k, F)$, we have

$$dL = \left(\frac{\partial L}{\partial h}\right)_F dh + \left(\frac{\partial L}{\partial k}\right)_F dk + \frac{\partial L}{\partial F}\, dF \tag{45}$$

From $F = f(h, k, \lambda)$, we have

$$dF = \frac{\partial F}{\partial h}\, dh + \frac{\partial F}{\partial k}\, dk + \frac{\partial F}{\partial \lambda}\, d\lambda \tag{46}$$

which can also be written as

$$dF = (a/r)(d\lambda - c_F\, dh + s_F\, dk) \tag{47}$$

because we have $\partial F/\partial h = -(a/r)c_F$, $\partial F/\partial k = (a/r)s_F$, and $\partial F/\partial\lambda = a/r$. Replacing dF in Eq. (45), we get

$$dL = \left[\left(\frac{\partial L}{\partial h}\right)_F - \frac{a}{r}c_F\frac{\partial L}{\partial F}\right]dh + \left[\left(\frac{\partial L}{\partial k}\right)_F + \frac{a}{r}s_F\frac{\partial L}{\partial F}\right]dk$$
$$+ \frac{a}{r}\frac{\partial L}{\partial F}\, d\lambda = \left(\frac{\partial L}{\partial h}\right)_{\text{tot}} dh + \left(\frac{\partial L}{\partial k}\right)_{\text{tot}} dk + \frac{a}{r}\frac{\partial L}{\partial F}\, d\lambda \tag{48}$$

where the partials $(\partial L/\partial h)_{\text{tot}}$ and $(\partial L/\partial k)_{\text{tot}}$ have been obtained in Ref. 9 as

$$\frac{\partial L}{\partial h} = \frac{(a^2/r^2)(1 - h^2 - k^2)^{\frac{1}{2}}\left[(1 - h^2\beta)s_L - hk\beta c_L\right] - s_L - 2(a/r)h}{hc_L - ks_L} \tag{49}$$

$$\frac{\partial L}{\partial k} = \frac{(a^2/r^2)(1 - h^2 - k^2)^{\frac{1}{2}}\left[(1 - k^2\beta)c_L - hk\beta s_L\right] - c_L - 2(a/r)k}{hc_L - ks_L} \tag{50}$$

Replacing dL in Eq. (48) in the identity given in Eq. (44) and equating the coefficients of all of the differentials, we get

$$\lambda_a^\lambda\, da + \lambda_h^\lambda\, dh + \lambda_k^\lambda\, dk + \lambda_p^\lambda\, dp + \lambda_q^\lambda\, dq + \lambda_\lambda^\lambda\, d\lambda$$
$$= \lambda_a^L\, da + \lambda_h^L\, dh + \lambda_k^L\, dk + \lambda_p^L\, dp + \lambda_q^L\, dq$$
$$+ \lambda_L^L\left[\left(\frac{\partial L}{\partial h}\right)_{\text{tot}} dh + \left(\frac{\partial L}{\partial k}\right)_{\text{tot}} dk + \frac{a}{r}\frac{\partial L}{\partial F}\, d\lambda\right]$$

and therefore

$$\lambda_a^\lambda = \lambda_a^L \tag{51}$$

$$\lambda_h^\lambda = \lambda_h^L + \lambda_L^L\left(\frac{\partial L}{\partial h}\right)_{\text{tot}} \tag{52}$$

$$\lambda_k^\lambda = \lambda_k^L + \lambda_L^L\left(\frac{\partial L}{\partial k}\right)_{\text{tot}} \tag{53}$$

$$\lambda_p^\lambda = \lambda_p^L \tag{54}$$

$$\lambda_q^\lambda = \lambda_q^L \tag{55}$$

$$\lambda_\lambda^\lambda = \lambda_L^L\frac{a}{r}\frac{\partial L}{\partial F} = \lambda_L^L\frac{a^2}{r^2}(1 - h^2 - k^2)^{\frac{1}{2}} \tag{56}$$

because $\partial L/\partial F = (a/r)(1 - h^2 - k^2)^{1/2}$ (Ref. 4). If the initial orbit is circular, then $(r)_0 = (a)_0$, $(h)_0 = (k)_0 = 0$, and therefore $(\lambda_L^L)_0 = (\lambda_\lambda^\lambda)_0$. Furthermore, if we optimize the initial and final orbital positions, we must require $(\lambda_L^L)_0$ and $(\lambda_L^L)_f = 0$. The first condition will in turn result in $(\lambda_h^\lambda)_0 = (\lambda_h^L)_0$ and $(\lambda_k^\lambda)_0 = (\lambda_k^L)_0$ such that the Lagrange multipliers of both formulations have identical initial values, namely, $(\lambda_a^L)_0 = (\lambda_a^\lambda)_0$, $(\lambda_h^L)_0 = (\lambda_h^\lambda)_0$, $(\lambda_k^L)_0 = (\lambda_k^\lambda)_0$, $(\lambda_p^L)_0 = (\lambda_p^\lambda)_0$, $(\lambda_q^L)_0 = (\lambda_q^\lambda)_0$, and $(\lambda_L^L)_0 = (\lambda_\lambda^\lambda)_0$.

We can verify that the mathematics presented in this paper are indeed correct by duplicating the minimum-time transfer example presented in Ref. 9. Given the initial and target orbits shown in Table 1, with both M_0 and M_f free, and using the acceleration value of $f_t = 9.8 \times 10^{-5}$ km/s^2, the converged values of the multipliers at time zero, as well as the optimized value of the initial mean longitude $(\lambda)_0 = -2.2747428$ rad and the minimum time $t_f = 58{,}089.9005$ s of the solution generated in Ref. 9 are used to run an open-loop trajectory with the present formulation to verify that the desired transfer is achieved. Starting from

$$\left(\lambda_a^L\right)_0 = 4.675229762\,\text{s/km} \qquad \left(\lambda_h^L\right)_0 = 5.413413947 \times 10^2\ \text{s}$$
$$\left(\lambda_k^L\right)_0 = -9.202702084 \times 10^3\ \text{s}$$
$$\left(\lambda_p^L\right)_0 = 1.778011878 \times 10^1\ \text{s} \tag{57}$$
$$\left(\lambda_q^L\right)_0 = -2.258455855 \times 10^4\ \text{s} \qquad (L)_0 = -2.274742851\,\text{rad}$$

and integrating both the system and adjoint equations until t_f, using the optimal control $\hat{u} = [\boldsymbol{\lambda}_z^T B^L(z)]^T/|\boldsymbol{\lambda}_z^T B^L(z)|$, we get the achieved orbit listed in Table 1, with $(\lambda_L^L)_f = 8.3054694 \times 10^{-3}$ s/rad and optimized $(L)_f = 19.6567242$ rad corresponding to $M_f = 46.169264$ deg. Note that $(L)_0$ in Eq. (57) corresponds to $M_0 = -130.333164$ deg, which is our initial optimized orbit location obtained in Ref. 9. The value of M_f achieved here is very close to the optimized solution of Ref. 9 of $M_f = 46.146408$ deg, duplicating thereby the optimal transfer example of Ref. 9.

We have also obtained an identical solution by iteration starting from guessed values for the various multipliers, $(\lambda)_0$, and flight time t_f. We now compare the multipliers' time histories of both formulations, namely, the (a, h, k, p, q, λ) formulation of Ref. 9

Table 1 Transfer parameters

Orbit	a, km	e	i, deg	Ω, deg	ω, deg	M, deg
Initial	7,000	0	28.5	0	0	−130.333164 (optimized)
Target	42,000	10^{-3}	1	0	0	Free
Achieved	41,999.9929	9.983×10^{-4}	0.999797	0.000326	359.995148	46.169264 (optimized)

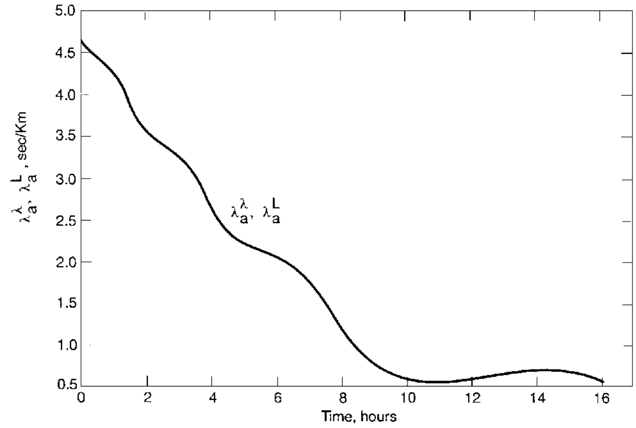

Fig. 1 Evolution of λ_a multipliers during low-Earth-orbit-to-near-GEO minimum-time transfer.

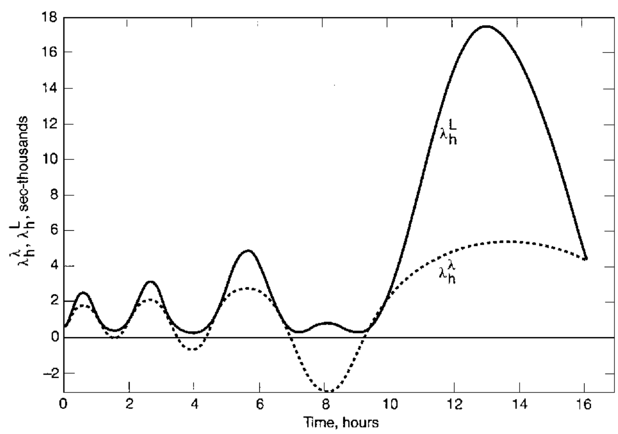

Fig. 2 Evolution of λ_h multipliers during optimal transfer.

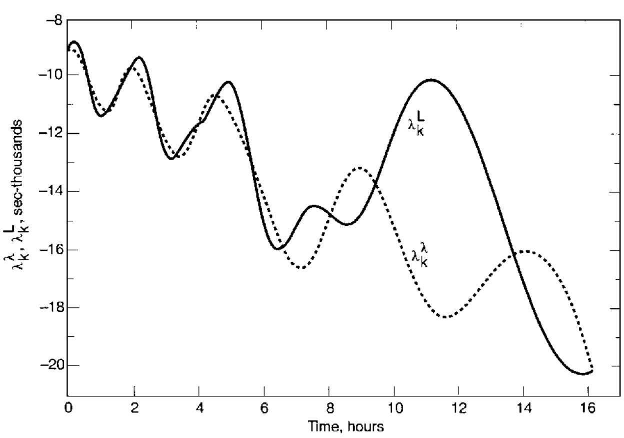

Fig. 3 Evolution of λ_k multipliers during optimal transfer.

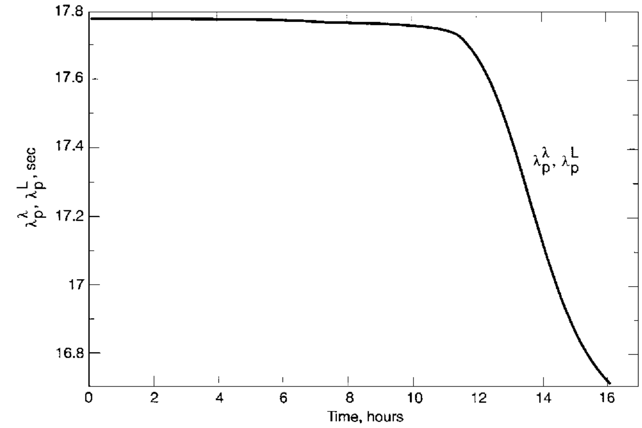

Fig. 4 Evolution of λ_p multipliers during optimal transfer.

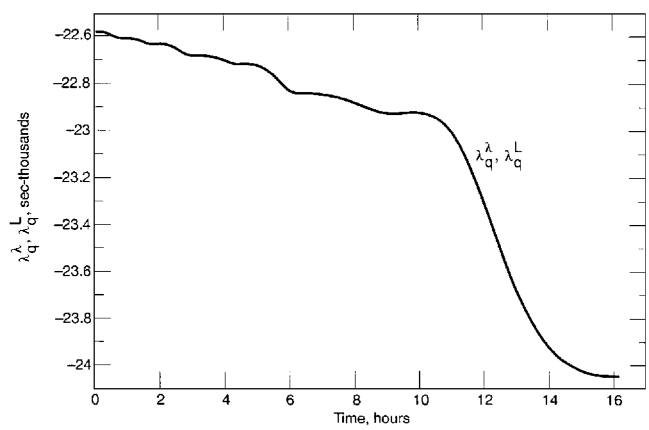

Fig. 5 Evolution of λ_q multipliers during optimal transfer.

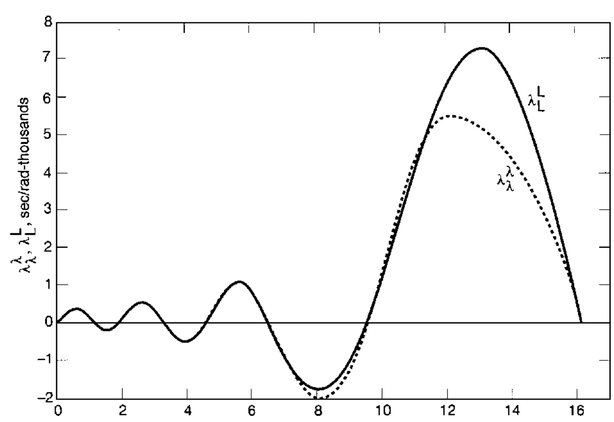

Fig. 6 Evolution of λ_L and λ_λ multipliers during optimal transfer.

and the present (a, h, k, p, q, L) formulation. Figure 1 shows the λ_a evolution with identical curves as shown in Eq. (51). Figures 2 and 3 are for the λ_h and λ_k multipliers with $(\lambda_h^\lambda)_0 = (\lambda_h^L)_0$, $(\lambda_k^\lambda)_0 = (\lambda_k^L)_0$, and $(\lambda_h^\lambda)_f = (\lambda_h^L)_f$, $(\lambda_k^\lambda)_f = (\lambda_k^L)_f$, as predicted by Eqs. (52) and (53) because $(\lambda_L^L)_0 = (\lambda_L^L)_f = 0$ to satisfy the boundary conditions at the initial and final times, which have been optimized for a free-free minimum-time transfer. Figures 4 and 5 show that $(\lambda_p^\lambda) = (\lambda_p^L)$ and $(\lambda_q^\lambda) = (\lambda_q^L)$ at all times, whereas Fig. 6 shows how $(\lambda_\lambda^\lambda)$ and (λ_L^L) evolve during the transfer, starting and ending at zero with essentially identical behavior during the first spiraling stage of the transfer where eccentricity stays small, and differing in magnitude as the eccentricity builds up for the direct transfer orbit to geostationary Earth orbit (GEO). This is expected because the true and mean longitudes are identical for zero eccentricity as is the case with the true and mean anomalies.

IV. Conclusions

The derivations leading to the generation of the full set of the equations of motion and of their corresponding adjoints are presented in terms of nonsingular orbit elements, with the true longitude representing the sixth state variable. Polar coordinates are used for the resolution of the perturbing accelerations, with all 12 differential equations written directly in terms of the true longitude itself. This formulation is simple in terms of both the boundary conditions and the complexity of the analytic form of the equations for the adjoints. This is amenable to the straightforward treatment of the Earth oblateness perturbation, with the various acceleration components written simply in terms of the true longitude. An example of a minimum-time, free-free, low-thrust orbit transfer generated earlier is duplicated by the present analysis to validate the mathematical derivations shown in this paper.

Appendix: B^L Matrix and Its Partial Derivatives

$$B_{11}^L = 2n^{-1}G^{-1}(ks_L - hc_L) \tag{A1}$$

$$\frac{\partial B_{11}^L}{\partial a} = -2n^{-2}\frac{\partial n}{\partial a}G^{-1}(ks_L - hc_L) \tag{A2}$$

$$\frac{\partial B_{11}^L}{\partial h} = 2n^{-1}hG^{-3}(ks_L - hc_L) - 2n^{-1}G^{-1}c_L \tag{A3}$$

$$\frac{\partial B_{11}^L}{\partial k} = 2n^{-1}kG^{-3}(ks_L - hc_L) + 2n^{-1}G^{-1}s_L \tag{A4}$$

$$\frac{\partial B_{11}^L}{\partial p} = \frac{\partial B_{11}^L}{\partial q} = 0 \tag{A5}$$

$$\frac{\partial B_{11}^L}{\partial L} = 2n^{-1}G^{-1}(kc_L + hs_L) \tag{A6}$$

$$B_{12}^L = 2n^{-1}ar^{-1}G \tag{A7}$$

$$\frac{\partial B_{12}^L}{\partial a} = -2n^{-2}ar^{-1}\frac{\partial n}{\partial a}G \tag{A8}$$

$$\frac{\partial B_{12}^L}{\partial h} = -2n^{-1}ar^{-2}\frac{\partial r}{\partial h}G - 2n^{-1}ar^{-1}hG^{-1} \tag{A9}$$

$$\frac{\partial B_{12}^L}{\partial k} = -2n^{-1}ar^{-2}G\frac{\partial r}{\partial k} - 2n^{-1}ar^{-1}kG^{-1} \tag{A10}$$

$$\frac{\partial B_{12}^L}{\partial p} = \frac{\partial B_{12}^L}{\partial q} = 0 \tag{A11}$$

$$\frac{\partial B_{12}^L}{\partial L} = -2n^{-1}ar^{-2}G\frac{\partial r}{\partial L} \tag{A12}$$

where $B_{13}^L = 0$ and therefore all of its partials are equal to zero;

$$B_{21}^L = -n^{-1}a^{-1}Gc_L \tag{A13}$$

$$\frac{\partial B_{21}^L}{\partial a} = -2^{-1}n^{-1}a^{-2}Gc_L \tag{A14}$$

$$\frac{\partial B_{21}^L}{\partial h} = n^{-1}a^{-1}hG^{-1}c_L \tag{A15}$$

$$\frac{\partial B_{21}^L}{\partial k} = n^{-1}a^{-1}kG^{-1}c_L \tag{A16}$$

$$\frac{\partial B_{21}^L}{\partial p} = \frac{\partial B_{21}^L}{\partial q} = 0 \tag{A17}$$

$$\frac{\partial B_{21}^L}{\partial L} = n^{-1}a^{-1}Gs_L \tag{A18}$$

$$B_{22}^L = n^{-1}a^{-2}rG^{-1}(h + s_L) + n^{-1}a^{-1}Gs_L \tag{A19}$$

$$\frac{\partial B_{22}^L}{\partial a} = 2^{-1}n^{-1}a^{-3}rG^{-1}(h + s_L) + 2^{-1}n^{-1}a^{-2}Gs_L \tag{A20}$$

$$\frac{\partial B_{22}^L}{\partial h} = n^{-1}a^{-2}G^{-1}(h + s_L)\left(\frac{\partial r}{\partial h} + rhG^{-2}\right) + n^{-1}a^{-2}rG^{-1} - n^{-1}a^{-1}hs_LG^{-1} \tag{A21}$$

$$\frac{\partial B_{22}^L}{\partial k} = n^{-1}a^{-2}G^{-1}(h + s_L)\left(\frac{\partial r}{\partial k} + rkG^{-2}\right) - n^{-1}a^{-1}ks_LG^{-1} \tag{A22}$$

$$\frac{\partial B_{22}^L}{\partial p} = \frac{\partial B_{22}^L}{\partial q} = 0 \tag{A23}$$

$$\frac{\partial B_{22}^L}{\partial L} = n^{-1}a^{-2}(h + s_L)G^{-1}\frac{\partial r}{\partial L} + n^{-1}a^{-2}rc_LG^{-1} + n^{-1}a^{-1}c_LG \tag{A24}$$

$$B_{23}^L = -n^{-1}a^{-2}rG^{-1}k(pc_L - qs_L) \tag{A25}$$

$$\frac{\partial B_{23}^L}{\partial a} = -2^{-1}n^{-1}a^{-3}rG^{-1}k(pc_L - qs_L) \tag{A26}$$

$$\frac{\partial B_{23}^L}{\partial h} = -n^{-1}a^{-2}G^{-1}k(pc_L - qs_L)\left(\frac{\partial r}{\partial h} + hrG^{-2}\right) \tag{A27}$$

$$\frac{\partial B_{23}^L}{\partial k} = -n^{-1}a^{-2}G^{-1}k(pc_L - qs_L)\left(\frac{\partial r}{\partial k} + krG^{-2}\right) - n^{-1}a^{-2}rG^{-1}(pc_L - qs_L) \tag{A28}$$

$$\frac{\partial B_{23}^L}{\partial p} = -n^{-1}a^{-2}rG^{-1}kc_L \tag{A29}$$

$$\frac{\partial B_{23}^L}{\partial q} = n^{-1}a^{-2}rG^{-1}ks_L \tag{A30}$$

$$\frac{\partial B_{23}^L}{\partial L} = -n^{-1}a^{-2}G^{-1}k(pc_L - qs_L)\frac{\partial r}{\partial L} + n^{-1}a^{-2}rG^{-1}k(ps_L + qc_L) \tag{A31}$$

$$B_{31}^L = n^{-1}a^{-1}Gs_L \tag{A32}$$

$$\frac{\partial B_{31}^L}{\partial a} = 2^{-1}n^{-1}a^{-2}Gs_L \tag{A33}$$

$$\frac{\partial B_{31}^L}{\partial h} = -n^{-1}a^{-1}G^{-1}hs_L \tag{A34}$$

$$\frac{\partial B_{31}^L}{\partial k} = -n^{-1}a^{-1}G^{-1}ks_L \tag{A35}$$

$$\frac{\partial B_{31}^L}{\partial p} = \frac{\partial B_{31}^L}{\partial q} = 0 \tag{A36}$$

$$\frac{\partial B_{31}^L}{\partial L} = n^{-1}a^{-1}Gc_L \tag{A37}$$

$$B_{32}^L = n^{-1}a^{-2}rG^{-1}(k + c_L) + n^{-1}a^{-1}Gc_L \tag{A38}$$

$$\frac{\partial B_{32}^L}{\partial a} = 2^{-1}n^{-1}a^{-3}rG^{-1}(k + c_L) + 2^{-1}n^{-1}a^{-2}Gc_L \tag{A39}$$

$$\frac{\partial B_{32}^L}{\partial h} = n^{-1}a^{-2}G^{-1}(k + c_L)\left(\frac{\partial r}{\partial h} + hrG^{-2}\right) - n^{-1}a^{-1}hG^{-1}c_L \tag{A40}$$

$$\frac{\partial B_{32}^L}{\partial k} = n^{-1}a^{-2}G^{-1}(k + c_L)\left(\frac{\partial r}{\partial k} + krG^{-2}\right) - n^{-1}a^{-1}kG^{-1}c_L + n^{-1}a^{-2}rG^{-1} \tag{A41}$$

$$\frac{\partial B_{32}^L}{\partial p} = \frac{\partial B_{32}^L}{\partial q} = 0 \tag{A42}$$

$$\frac{\partial B_{32}^L}{\partial L} = n^{-1}a^{-2}G^{-1}(k + c_L)\frac{\partial r}{\partial L} - n^{-1}a^{-2}rG^{-1}s_L - n^{-1}a^{-1}Gs_L \tag{A43}$$

$$B_{33}^L = n^{-1}a^{-2}rG^{-1}h(pc_L - qs_L) \tag{A44}$$

$$\frac{\partial B_{33}^L}{\partial a} = 2^{-1}n^{-1}a^{-3}rG^{-1}h(pc_L - qs_L) \tag{A45}$$

$$\frac{\partial B_{33}^L}{\partial h} = n^{-1}a^{-2}G^{-1}h(pc_L - qs_L)\left(\frac{\partial r}{\partial h} + hrG^{-2}\right) + n^{-1}a^{-2}rG^{-1}(pc_L - qs_L) \tag{A46}$$

$$\frac{\partial B_{33}^L}{\partial k} = n^{-1}a^{-2}G^{-1}h(pc_L - qs_L)\left(\frac{\partial r}{\partial k} + krG^{-2}\right) \tag{A47}$$

$$\frac{\partial B_{33}^L}{\partial p} = n^{-1}a^{-2}rG^{-1}hc_L \tag{A48}$$

$$\frac{\partial B_{33}^L}{\partial q} = -n^{-1}a^{-2}rG^{-1}hs_L \tag{A49}$$

$$\frac{\partial B_{33}^L}{\partial L} = n^{-1}a^{-2}G^{-1}h(pc_L - qs_L)\frac{\partial r}{\partial L} - n^{-1}a^{-2}rG^{-1}h(ps_L + qc_L) \tag{A50}$$

$$B_{43}^L = 2^{-1}n^{-1}a^{-2}rG^{-1}Ks_L \tag{A51}$$

$$\frac{\partial B_{43}^L}{\partial a} = 4^{-1}n^{-1}a^{-3}rG^{-1}Ks_L \tag{A52}$$

$$\frac{\partial B_{43}^L}{\partial h} = 2^{-1}n^{-1}a^{-2}G^{-1}Ks_L\left(\frac{\partial r}{\partial h} + hrG^{-2}\right) \tag{A53}$$

$$\frac{\partial B_{43}^L}{\partial k} = 2^{-1}n^{-1}a^{-2}G^{-1}Ks_L\left(\frac{\partial r}{\partial k} + krG^{-2}\right) \tag{A54}$$

$$\frac{\partial B_{43}^L}{\partial p} = n^{-1}a^{-2}rG^{-1}ps_L \tag{A55}$$

$$\frac{\partial B_{43}^L}{\partial q} = n^{-1}a^{-2}rG^{-1}qs_L \tag{A56}$$

$$\frac{\partial B_{43}^L}{\partial L} = 2^{-1}n^{-1}a^{-2}G^{-1}Ks_L\frac{\partial r}{\partial L} + 2^{-1}n^{-1}a^{-2}rG^{-1}Kc_L \tag{A57}$$

$$B_{53}^L = 2^{-1}n^{-1}a^{-2}rG^{-1}Kc_L \tag{A58}$$

$$\frac{\partial B_{53}^L}{\partial a} = 4^{-1}n^{-1}a^{-3}rG^{-1}Kc_L \tag{A59}$$

$$\frac{\partial B_{53}^L}{\partial h} = 2^{-1}n^{-1}a^{-2}G^{-1}Kc_L\left(\frac{\partial r}{\partial h} + hrG^{-2}\right) \tag{A60}$$

$$\frac{\partial B_{53}^L}{\partial k} = 2^{-1}n^{-1}a^{-2}G^{-1}Kc_L\left(\frac{\partial r}{\partial k} + krG^{-2}\right) \tag{A61}$$

$$\frac{\partial B_{53}^L}{\partial p} = n^{-1}a^{-2}rG^{-1}pc_L \tag{A62}$$

$$\frac{\partial B_{53}^L}{\partial q} = n^{-1}a^{-2}rG^{-1}qc_L \tag{A63}$$

$$\frac{\partial B_{53}^L}{\partial L} = 2^{-1}n^{-1}a^{-2}G^{-1}Kc_L\frac{\partial r}{\partial L} - 2^{-1}n^{-1}a^{-2}rG^{-1}Ks_L \tag{A64}$$

$$B_{63}^L = na^2r^{-2}G + n^{-1}a^{-2}rG^{-1}(qs_L - pc_L) \tag{A65}$$

$$\frac{\partial B_{63}^L}{\partial a} = 2^{-1}n^{-1}a^{-3}rG^{-1}(qs_L - pc_L) \tag{A66}$$

$$\frac{\partial B_{63}^L}{\partial h} = n^{-1}a^{-2}(qs_L - pc_L)G^{-1}\left(\frac{\partial r}{\partial h} + rhG^{-2}\right) \tag{A67}$$

$$\frac{\partial B_{63}^L}{\partial k} = n^{-1}a^{-2}(qs_L - pc_L)G^{-1}\left(\frac{\partial r}{\partial k} + rkG^{-2}\right) \tag{A68}$$

$$\frac{\partial B_{63}^L}{\partial p} = -n^{-1}a^{-2}rG^{-1}c_L \tag{A69}$$

$$\frac{\partial B_{63}^L}{\partial q} = n^{-1}a^{-2}rG^{-1}s_L \tag{A70}$$

$$\frac{\partial B_{63}^L}{\partial L} = n^{-1}a^{-2}(qs_L - pc_L)G^{-1}\frac{\partial r}{\partial L} + n^{-1}a^{-2}r(qc_L + ps_L)G^{-1} \tag{A71}$$

Finally,

$$\frac{\partial}{\partial a}\left[\frac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2}\right] = -\frac{3}{2}nar^{-2}G \tag{A72}$$

$$\frac{\partial}{\partial h}\left[\frac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2}\right] = -2na^2r^{-3}G\frac{\partial r}{\partial h} - na^2r^{-2}hG^{-1} \tag{A73}$$

$$\frac{\partial}{\partial k}\left[\frac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2}\right] = -2na^2rG\frac{\partial r}{\partial k} - na^2r^{-2}kG^{-1} \tag{A74}$$

$$\frac{\partial}{\partial L}\left[\frac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2}\right] = -2na^2r^{-3}G\frac{\partial r}{\partial L} \tag{A75}$$

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